Non-archimedean bases of topological spaces

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Abstract. Non-Archimedean bases are essential for defining and studying topologies that can be metrized using non-Archimedean metrics. A topological space is non-Archimedean metrizable if it admits a topology derived from a non-Archimedean metric, a metric satisfying the strong triangle inequality. This paper examines the role of non-Archimedean bases in establishing the necessary and sufficient conditions for a topological space to be non-Archimedean metrizable. Furthermore, it presents the non-Archimedean property in zero-dimensional topological spaces, emphasizing bases composed entirely of clopen (simultaneously open and closed) sets.

ملخص: الاساسات الغير ارشيميدية ضرورية لتعريف ودراسة التبولوجيات القابلة للمترية الغير ارشيميدية. الفضاء التبولوجي يكون قابل للمترية الغير ارشيميدية اذا تطابق مع التبولوجي المشتق من المترية الغير ارشيميدية، أي المترية التي تحقق المتباينة المثلثية القوية. تتناول هذه الورقة البحثية دور الاساسات الغير ارشيميدية في تأسيس الشروط الضرورية والكافية للفضاء ليكون قابل للمترية الغير ارشيميدية. علاوة على ذلك فهي تعرض الخاصية الغير ارشيميدية في الفضاءات التبولوجية ذات البعد الصفري مع التركيز على الاساسات المكونة بالكامل من مجموعات مفتوحة ومغلقة في نفس الوقت.

Keywords. Compact space, metric, metrizable, non-archemedean, non-archimedean base, normal, topological space, uniform base, zero-dimension space.

1 Introduction

The concept of non-Archimedean bases in topological spaces traces its origins to Kurt Hensel's introduction of p-adic numbers, which established a topology governed by the ultrametric inequality. These spaces are characterized by clopen (simultaneously open and closed) balls, leading to a disconnected structure that challenges classical intuition. Alexander Grothendieck further developed their utility in rigid analytic geometry, especially for the study of algebraic varieties. Today, non-Archimedean spaces play a pivotal role in Berkovich spaces, tropical geometry, and dynamical systems, creating profound connections between number theory, geometry, and physics in modern mathematics.

The non-Archimedean base of a topological space consists of open sets closed under finite intersections, essential for understanding non-Archimedean, metrizable, and zero-dimensional spaces. A space is non-Archimedean metrizable if it has a topology generated by a non-Archimedean metric that satisfies the ultrametric inequality. This results in unique topological properties, such as highly disconnected spaces with clopen sets, setting non-Archimedean metric spaces apart from traditional ones. These bases are closely related to zero-dimensional spaces, where clopen sets form a basis, and open balls in non-Archimedean spaces are clopen, making them natural examples of zero-dimensional spaces. Thus, non-Archimedean bases provide a concrete and rich framework for understanding and working with zero-dimensional spaces.

This paper examines the role of non-Archimedean bases in general topological spaces, non-Archimedean metrizable spaces, and zero-dimensional spaces. Section 4 presents Theorems 4.2, 4.3, 4.4, and 4.10, which outline the properties of these bases, demonstrating how they can characterize non-Archimedean metrizable spaces and reveal their topological structure.

2 Definitions and preliminaries

Let $\mathcal{K}$ be a family of subsets of a set X. Then $\mathcal{K}$ has *rank zero*, if for any pair $K_1, K_2 \in \mathcal{K}$ with non-empty intersection, we have either $K_1 \subset K_2$ or $K_2 \subset K_1$.

Let (X, d) be a metric space. We call d *a non-archimedean metric*, n. - a. metric, (Ultra-metric) if d satisfies the strong triangular inequality $d(x,y) \leq \max\{d(x,z), d(z,y)\}$, where $x, y, z \in X$. For each $x \in X$ and $\epsilon > 0$, define the set $B_\epsilon(x) = (y \in X : d(x,y) < \epsilon)$ to be an open ball with radius ϵ and center x. In this case, we call (X, d) a non-archimedean metric (ultrametric) space.

Proposition 2.1 *Let (X,d) be a metric space and d be n.-a. metric on X. Then the balls $\{B_\varepsilon(x): x \in X, \varepsilon > 0\}$ form a base of rank zero.*

Proof. To show that, if $B_\varepsilon(x) \cap B_\delta(y) \neq \emptyset$ then $B_\varepsilon(x) \subset B_\delta(y)$ or $B_\delta(y) \subset B_\varepsilon(x)$. Suppose that $\varepsilon < \delta$ and $B_\varepsilon(x) \cap B_\delta(y) \neq \emptyset$, then there exist $z \in B_\varepsilon(x) \cap B_\delta(y)$ so $d(x,z) < \varepsilon$ and $d(y,z)) < \delta$. Let $w \in B_\varepsilon(x)$ then $d(x,w) < \varepsilon$. If $d(y,w) \leq \max\{d(y,x), d(x,w)\}$ so either $d(y,w) \leq d(x,w)$, and then $d(y,w) < \varepsilon < \delta$, hence $w \in B_\delta(y)$ Or $d(y,w) \leq d(y,x)$, so that $d(y,w) \leq \max\{d(y,z), d(z,x)\}$. If $d(y,w) \leq d(y,z)$, and then $d(y,w) < \delta$, so $w \in B_\delta(y)$. If $d(y,w) \leq d(x,z)$ then $d(y,w) < \varepsilon < \delta$. Hence $w \in B_\delta(y)$. So from all the previous cases, we can say that, if two open balls intersect, then one (that of a smaller radius) is contained in the other.

A base of a space X is called *a uniform base*, if for each $x \in X$ and each open subset U of X contains x, only a finite number of basis sets contain x and interest U^c.

Proposition 2.2 *Any metric space has a uniform base.*

Proof. Let X be a metric space, and let d be a metric on X.

Let $\mathcal{B} = \left\{B_{\frac{1}{n}}(x): x \in X, n \in \mathbb{N}\right\}$, then $\mathcal{B}$ is a base of X . To show that $\mathcal{B}$ is a uniform base of X. Let $x \in X$ and U be an open set containing x , then there exist n_0 such that $B_{\frac{1}{n_0}}(x) \subset U$. Thus only balls with radius $\frac{1}{m}(m = 1,2,\dots,n_0 - 1)$ can contain x and intersect U^c. Hence $\mathcal{B}$ is a uniform base of X.

A cover $\mathcal{U}$ of a space X is *a refinement* of another cover $\mathcal{V}$ of the same space X, in other words $\mathcal{U}$ refines $\mathcal{V}$, if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$.

A collection $\mathcal{H}$ of a space X is called *locally finite* if, for each $x \in X$, there exists an open neighbourhood V of x such that V intersects only finitely many elements of $\mathcal{H}$. The collection $\mathcal{H}$ is called σ-locally finite if it can be expressed as a countable union of locally finite collections.

A *paracompact space* X is defined as a Hausdorff in which every open cover of X has a locally finite open refinement.

The following proposition gives a necessary condition for a paracompact space to be metrizable. For the proof of this theorem, see [1].

Proposition 2.3 *Any paracompact space with a uniform base is metrizable.*

Let X be a topological space. A subset A of X is called G_δ - *set* if there exists a countable collection of open sets $\{U_i\}_{i=1}^{\infty}$ such that $A = \bigcap_{i=1}^{\infty} U_i$. A subset F of X is called *zero-set* if there exists a continuous function $f: X \longrightarrow I$ such that $F = f^{-1}(\{0\})$.

The next theorem is related to the characterization of zero-sets and G_δ - sets in topology, particularly in the context of normal spaces. Here's a formal version of the statement:

Theorem 2.4 *Let X be a normal space and A be a subset of X. Then A is a closed G_δ-set if and only if A is a zero-set.*

Proof. (⇒) Let A be a closed G_δ-set in a normal space X, then the complement of A is an F_σ-set. Hence $X \backslash A = \bigcup_{i=1}^{\infty} C_i$, where C_i is a closed subset of X for each $i \in N$. By Urysohns lemma, for each $i \in IN$ there exists a continuous function $f_i: X \to I$ such that $f_i(x) = \{0\}$ for $x \in A$ and $f_i(x) = \{1\}$ for $x \in C_i$. Let $g: X \to I$ defined by $g(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x)$ for each $x \in X$, then g is a continuous function. For each $x \in A$ we have $g(x) = \{0\}$, and if $x \notin A$ there exists an i such that $x \in C_i$, and $g(x) \geq \frac{1}{2^i} f_i(x) = \frac{1}{2^i} > 0$, so $A = g^{-1}(\{0\})$.

(⇐) The one point set $\{0\} \subset I$, is a closed G_δ-set. Let $f: X \to I$ be a continuous function, such that $A = f^{-1}(\{0\})$. Then A is a closed G_δ-set in X.

Proposition 2.5 *If A is a closed subset of a metrizable space X, then A is a G_δ - set.*

Proof. Let A be a closed subset of a metrizable space X. Let d be a metric on the set X, by Theorem 2.4, we need to show that A is a zero-set. Since $A = \bar{A}$ and $\bar{A} = \{x: d(x, A) = 0\}$, let $h(x) = d(x, A)$. So $h(A) = \{0\}$, Hence $A = h^{-1}(\{0\})$.

A perfectly normal space is a normal space where every closed set in the space is a G_δ - set.

Proposition 2.6 *Any metrizable space is perfectly normal.*

Proof. Let X be a metrizable space and K be a closed subset of X. Then X is a normal space. We only need to show that K is a G_δ - set in X. For each $n \in \mathbb{N}$, define open sets $U_n = \{x \in X: d(x,K) < 1/n\}$, where $d(x,K) = inf\{d(x,y): y \in K\}$. As $n \longrightarrow \infty$, U_n shrinking toward K. To show that $K = \bigcap_{n=1}^{\infty} U_n$. Let $x \in K$, then $d(x,K) = 0$. So $x \in U_n$ for all n which implies that $x \in \bigcap_{n=1}^{\infty} U_n$. Now let $x \in \bigcap_{n=1}^{\infty} U_n$, then for each n, $d(x,K) < 1/n$. Since K is closed, so $d(x,K) = 0$ and it means that $x \in K$. Therefore, $K = \bigcap_{n=1}^{\infty} U_n$, which shows that K is $G_\delta - set$.

3 Non-archimedean topological spaces

Definition 3.1 *A T_1 - space X is said to be a non-archimedean (n. - a.) space if X has a base of rank zero. In this case, we call the base of rank zero a non-archimedean ($n.-a$. base).* A subset U of a space X is called a clopen set if it is both closed and open simultaneously.

Lemma 3.2 *All members of any n. - a. base are clopen.*

Proof. Let $\mathcal{B}$ be a n. - a. base of a space X and let $B \in \mathcal{B}$. To show that B is closed. If $x \in \bar{B}$ and $x \notin B$, then every basic neighbourhood B^* of x intersect B. So either $B^* \subset B$, in this case, $x \in B$, gives a contradiction. Or B is contained in every basic neighbourhood B^* containing x. So $B \subset (\cap\{B^*: B^* \in \mathcal{B}, x \in B^*\})$. But X is a $T_1 -$ space, hence $\cap\{B^*: B^* \in \mathcal{B}, x \in B^*\} = \{x\}$, which gives a contradiction.

Every subspace of a n. - a. space X is n. - a., since every subspace of a T_1 - space is T_1 and if $\mathcal{B}$ is a base of X has rank zero then it's trace is a base of rank zero.

A family $\mathcal{A}$ of subsets of a space X is called *discrete* if for any point $x \in X$, there exists an open set U of x that intersects at most one element of $\mathcal{A}$. This means that every element of the family $\mathcal{A}$ is isolated from the others.

Proposition 3.3 *Let X be a n. - a. space, and $\mathcal{B}$ is a n. - a. base of X, then any locally finite collection of disjoint basic sets is a discrete family.*

Proof. Let $x \in X$ and let $\mathcal{S} = \{B_\alpha : \alpha \in \Gamma\}$ be a disjoint locally finite subcollection of $\mathcal{B}$. If there exists $\beta \in \Gamma$ such that $x \in B_\beta$, then B_β is a neighbourhood of x and does not intersect any member of $\mathcal{S}$. If $x \notin \bigcup_{\alpha \in \Gamma} B_\alpha$, since $\mathcal{S}$ is a locally finite closed collection, then $\bigcup_{\alpha \in \Gamma} B_\alpha$ is closed, so $(\bigcup_{\alpha \in \Gamma} B_\alpha)^c$ is open containing x and does not intersect any member of $\mathcal{S}$, hence $\mathcal{S}$ is a discrete family.

A chain of a family $\mathcal{F}$ of subsets of a space X is a subcollection $\mathcal{C}$ of $\mathcal{F}$ such that for any two sets $C_1, C_2 \in \mathcal{C}$ either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$. In other words, any two sets of $\mathcal{C}$ are comparable under inclusion.

Lemma 3.4 *Let X be a n. - a. space and $\mathcal{B}$ is a n. -a. base of X, then the union of any chain in $\mathcal{B}$ is a clopen subset of X. Moreover, the set of all unions of chains in $\mathcal{B}$ is a n. - a. base of X.*

Proof. Let $\mathcal{C}$ be any chain in $\mathcal{B}$ and let $D = \cup\{B_\alpha : B_\alpha \in \mathcal{C}\}$.

Firstly, to show that D is clopen. For any $x \in X$ and $x \notin D$ and $B_\alpha \in \mathcal{C}$, there exist a basic element $B \in \mathcal{B}$ such that $x \in B$ and $B \cap B_\alpha = \emptyset$. If $B \cap B_\beta \neq \emptyset$ for some $B_\beta \in \mathcal{C}$, then either $B \subset B_\beta$, hence $x \in B_\beta$, so we have $x \in D$, which gives a contradiction. Or $B_\beta \subset B$, so $B_\beta \cap B_\alpha = \emptyset$, which gives a contradiction. Hence $B \cap B_\alpha = \emptyset$ for each $B_\alpha \in \mathcal{C}$. That is mean $B \cap D = \emptyset$, thus $B \subset D^c$, so D is clopen.

Secondly, to show that the set of all unions of chains in $\mathcal{B}$ is a $n.-a.$ base of X. If $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, and so there exists a chain $\mathcal{C}$ in $\mathcal{B}$ such that $B \in \mathcal{C}$. So $x \in C$, for some C. Now, If D_1, D_2 are any two such union of two chains C_1, C_2 in $\mathcal{B}$ then

$$D_1 \cap D_2 = \{\cup\{B_\alpha : B_\alpha \in C_1\}\} \cap \left\{\cup\{B_\beta : B_\beta \in C_2\}\right\}$$
$$= \cup\left\{B_\alpha \cap B_\beta : B_\alpha \in C_1, B_\beta \in C_2\right\}.$$

Hence either $B_\alpha \cap B_\beta = \emptyset$ for each $B_\alpha \in C_1$. and $B_\beta \in C_2$. In this case $D_1 \cap D_2 = \emptyset$. Or $B_\alpha \cap B_\beta \neq \emptyset$ for some $B_\alpha \in C_1, B_\beta \in C_2$ so either $D_1 \subset D_2$ or $D_2 \subset D_1$.

The following theorem captures a key structural property of non-Archimedean spaces, reflecting their unique topological nature. The existence of a non-Archimedean base ensures that the topology of the space is entirely determined by these clopen subsets. The characterization of clopen sets as unions of locally finite, disjoint subcollections of the non-archimedean base emphasizes the highly disconnected nature of the space, where clopen sets serve as the fundamental building blocks of its topology.

Theorem 3.5 [9] *Let X be a n.-a. space. Then there is a n. -a. base $\mathcal{B}^*$ of X, such that the subset A of X is clopen if and only if A is a union of a locally finite disjoint subcollection of $\mathcal{B}^*$*

Proof. ($\Rightarrow$) Let A be a clopen subset of X. For each $x \in A$, let $B_x = \cup\{B_\alpha : B_\alpha \in \mathcal{B}^*, x \in B_\alpha \subset A\}$ where $\mathcal{B}^*$ is a n . -a . base of X. Then $\mathcal{C} = \{B_x : x \in A\}$ is a clopen partition of A and a discrete collection; since for each $x \in X$, if $x \in A, B_x$ is a neighbourhood of x which does not intersect any other member of $\mathcal{C}$. If $x \notin A$, since A is closed, so there exist $B_\alpha \in B$ with $x \in B_\alpha, B_\alpha \cap A = \varnothing$.

($\Leftarrow$) Let $\mathcal{B}^*$ be a n. -a . base of X , and let $\mathcal{C}$ be the unions of all chains in $\mathcal{B}^*$. Then by lemma 3.4, $\mathcal{C}$ is a n.-a. base of X and all $C_\alpha \in \mathcal{C}$ are clopen. So for each locally finite collection $\{C_\alpha : C_\alpha \in \mathcal{C}, \alpha \in \Gamma\}$ we have $A = \bigcup_{\alpha\in\Gamma} C_\alpha = \overline{\bigcup_{\alpha\in\Gamma} C_\alpha} = \overline{A}$ is clopen.

Example: Let $\mathcal{N}$ be the discrete topology on the set of natural numbers $\mathbb{N}$. Let $\mathbb{N}^* = \mathbb{N} \cup \alpha$ and let $\mathcal{N}^*$ be a topology on $\mathbb{N}^*$. All subsets of $\mathbb{N}$ are open in $(\mathbb{N}^*, \mathcal{N}^*)$, if $U \subseteq \mathbb{N}^*$ and $\alpha \in U$, then U is open in $(\mathbb{N}^*, \mathcal{N}^*)$ if and only if $\mathbb{N} - U$ is a compact in $(\mathbb{N}, \mathcal{N})$. This example shows that in a n. - a. space, not any union of discrete sets of a n. - a. base is clopen; $N = \bigcup_{n\in N}\{n\}$, is not closed in N^*. Therefore, locally finiteness cannot be deleted in the last theorem.

The proof of the next proposition is in [9].

Proposition 3.6 *Any n. - a. space is hereditarily ultra-paracompact.*

Note that: if X is a hereditary ultra-paracompact space, then it is not necessary a n. - a. space.

For example; Let $\mathbb{R}$ be the set of real numbers and $\mathfrak{J}$ be the family of all intervals $[a, b)$ where $a, b \in \mathbb{R}$, $a < b$. Then the members of $\mathfrak{J}$ are clopen with respect to the topology generated by $\mathfrak{J}$ on $\mathbb{R}$. This topology is called “Sorgenfrey-line” and denotes it by $\mathbb{R}_S$. The sorgenfrey-line $\mathbb{R}_S$ is hereditarily ultra-paracompact, since $\mathbb{R}_S$ is hereditarily Lindelof, but $\mathbb{R}_S$ is not a n. - a. space.

In non-Archimedean spaces, having a countable dense subset (separability) often leads to second countability. This is because the topology of a non-Archimedean space is defined by its clopen sets, which serve as a base. When the space is separable, it is possible to construct a countable clopen base that reflects the separable structure. The next theorem states these relationships.

Theorem 3.7 *A n.-a. space is separable if and only if it is second countable.*

Proof. Every second countable space is separable, so in particular, if a non-Archimedean space is second countable, it is automatically separable. Let X be a separable non-archimedean space. Thus X has a countable dense subset $D=\{x_1,x_2,...\}$. The intersections of the clopen sets with D (or those defined around points in D) often suffice to construct a countable base.

4 Applications of non-archimedean bases

Non-Archimedean bases are fundamental tools in topology, especially in the study of non-Archimedean metrizable spaces and zero-dimensional spaces. These concepts have deep implications in many areas of topology. Below is a detailed discussion of some of their applications:

4.1 Non-archimedean merizability property

A natural question arises: under what conditions can a topological space be classified as non-Archimedean metrizable? Specifically, what criteria allow us to describe the topological structure of such spaces by defining an appropriate non-Archimedean metric? In this section, we present important theorems that answer this question.

Definition 4.1 *A space X is called n. - a. metrizable (ultrametrizable) if there exists a n. - a. metric d on X such that the topology induced by d coincides with the original topology on X.*

The next theorem characterizes topological spaces to be *n. – a.* metrizable and can be found in books on general topology.

Theorem 4.2 *A topological space X is n. - a. metrizable if and only if there exists a σ - locally finite clopen base of X.*

Proof. (⇒) Let X be a n. – a. metrizable space, and let

$U_n = \left\{B_{\frac{1}{n}}(x): x \in X\right\}$. Since any two members of U_n are disjoint or identical, so U_n is a locally finite clopen collection for all n. Let $\mathrm{U} = \bigcup_{n=1}^{\infty} \mathrm{U}_n$, then U be a σ-locally finite clopen base of X.

($\Leftarrow$) Let $\{\mathrm{U}_{n\alpha}\}_{\substack{n\in\in N\\ \alpha\in\Gamma}}$ be a countable number of a locally finite clopen base of X be given. Let β be the cardinal of the set $\{\gamma\}$ of all possible indices $\gamma = n\alpha$, we put

$$U_{n(m\alpha)} = \begin{cases} U_{n\alpha}, & \text{if} n = m \\ \varnothing, & \text{if} n \neq m \end{cases}$$

The families $\{\mathrm{U}_{n\alpha}\}_{\substack{n\in IN\\ \alpha\in\Gamma}}$ remain locally finite. We define for each $\mathrm{x} \in \mathrm{X}$ and each pair $\mathrm{n}\gamma$ a function $\zeta_{n\gamma}$ such that

$$\zeta_{n\gamma}(\mathrm{x}) = \begin{cases} 1, & \text{if} \quad x \in U_{n\gamma} \\ 0, & \text{if} \quad x \notin U_{n\gamma} \end{cases}$$

Let $f: \mathrm{X} \to \mathrm{N}(\beta)$ be a function defined by $f(\mathrm{x}) = \left\{\zeta_{n\gamma}(\mathrm{x})\right\}$, for each $\mathrm{x} \in \mathrm{X}$. To show that f is an embedding. The mapping f is one-to-one, since to each pair of different points x and y in X, there corresponds a $\mathrm{U}_{n\alpha}$ containing x and not containing y, therefore $\zeta_{n\alpha}(\mathrm{x}) = 1$, $\zeta_{n\alpha}(\mathrm{y}) = 0$, hence $f(\mathrm{x}) \neq f(\mathrm{y})$. The map f is continuous, since if G_ε is any ε-neighbourhood with $\varepsilon = \frac{1}{m}$ (m sufficiently large) in $N(\beta)$ of a point $f(\mathrm{x}) = \{\zeta_{n\alpha}(\mathrm{x})\}$. If $n \leqslant m$, there is only a finite number of $\mathrm{U}_{n\alpha}$ which intersect a certain neighbourhood $G_\varepsilon(x)$ of x. there are two types of $\mathrm{U}_{n\alpha}$, one of them $\mathrm{U}_{k\alpha}$ which contains x, and the other $\mathrm{U}_{l\alpha}$ is not containing x. Let $\mathrm{V}(\mathrm{x}) = (\bigcap_{-k<m} \mathrm{U}_{k\alpha} \setminus \bigcup_{l<m} \mathrm{U}_{l\alpha}) \cap \mathrm{G}_\varepsilon(\mathrm{x})$ then $\mathrm{V}(\mathrm{x})$ is an open neighbourhood of x. To show $f(\mathrm{V}(\mathrm{x})) \subset \mathrm{G}_\varepsilon(\mathrm{x})$, let $\mathrm{y} \in \mathrm{V}(\mathrm{x})$ and $\mathrm{n} \leq \mathrm{m}$, $\zeta_{n\alpha}(\mathrm{x}) - \zeta_{n\alpha}(\mathrm{y}) = 0$, since x and y are in $\mathrm{U}_{n\alpha}$ or in it's complement. Now, let ρ be a n. - a. metric on $N(\beta)$ as described in lemma (4.2.4). For any $f(\mathrm{x}), f(\mathrm{y}) \in \mathrm{N}(\beta)$ then

$$\begin{aligned} \rho(f(\mathrm{x}), f(\mathrm{y})) &= \rho(\zeta_{n\alpha}(\mathrm{x}), \zeta_{n\alpha}(\mathrm{y})) \\ &= \max_{n\alpha}\left\{\frac{1}{n}\left|\zeta_{n\alpha}(\mathrm{x}) - \zeta_{n\alpha}(\mathrm{y})\right|\right\} \leq \frac{1}{m+1} \end{aligned}$$

The map f is open, since, if $\mathrm{H} \subset \mathrm{X}$ is an open set and $\mathrm{x} \in \mathrm{H}$, there exists $U_{j\alpha}$ With $x \in U_{j\alpha} \subset H$. Hence $\zeta_{j\alpha}(x) = 1$. If $\rho(f(x), f(y)) < \frac{1}{j}$ is fulfilled for a certain point $y \in X$. It follows that $\zeta_{j\alpha}(y) = 1$ which implies that $y \in U_{j\alpha} \subset H$. The set $f(H)$ is therefore open in $f(X)$. The mapping f being an embedding, thus it induces the required $n.-a$. metric on X by considering $f(X)$ instead of X.

Theorem 4.3 *A space X is n. - a. metrizable if and only if X has a n. - a. uniform base.*

Proof. (⇒) Let X be a n. - a. metrizable space, then there exists a n. - a. metric d on X such that the topology induced by d coincides with the topology on X, let $B = \left\{B_{\frac{1}{n}}(x) : x \in X, n \in \mathbb{N}\right\}$, then B is a n. - a. base and by Proposition 2.2, B is a uniform base of X .

(⇐) Let X be a topological space and X has a n. - a. uniform base, so X is n. - a. space. Hence by Peopositionmetricuniform X is paracompact space and by Proposition 3.6, X is metrizable n. - a. space, so X is n. - a. metrizable space.

The next theorem is in [9] and it characterizes when a compact space possesses the non-Archimedean metrizability property. In such spaces, the topology is determined by a base of clopen sets. Compactness ensures that this clopen base is finite at small scales, which is consistent with the structure of a non-Archimedean metric.

Theorem 4.4 *A compact space is n. - a. metrizable if and only if it has a n. - a. base.*

Proof. (⇒) Let X be a n.-a. metrizable, so X has a n. - a. base.

(⇐) Let $\mathcal{B}$ be a n. - a. base of a compact space X, to show that X is n. - a. metrizable. By Theorem 4.3, it is enough to show that X has a n. -a. uniform base. Let $\mathbf{B}(x) = \{B_\alpha \in \mathcal{B} : x \in B_\alpha, \alpha \in \Gamma\}$, then $\mathbf{B}(x)$ is totally ordered by the n. - a. property of $\mathcal{B}$, and because of the compactness of $X, B(x)$ well ordered by $B_\alpha < B_\beta$ if and only if $B_\alpha \supset B_\beta$, since by lemma 3.4, $\bigcup_{\alpha \in I} B_\alpha$ is clopen and hence compact, thus there is a greatest $B_{\alpha_n} \in B(x)$ where $B_{\alpha_n} = \bigcup_{\alpha \in \Gamma} B_\alpha$. For any $\alpha \in \Gamma$ let $B_{\alpha+1}$ be the greatest set in $B(x)$ among all that are contained in B_α. Now, let U be an open neighborhood of x. To show that only finitely many members of $\mathbf{B}(x)$ intersect $X\backslash U$. suppose not; that is there are infinitely many $B_{\alpha_n} \in \mathbf{B}(x)$ intersecting $X\backslash U$. Then there exists a sequence $B_1 \supset B_2 \supset \cdots \supset B_n \supset \cdots$, of sets in $B(x)$ and a sequence $\{x_n\}_{n=1}^{\infty}$ of points such that $x_n \in X\backslash U$ and $x_n \in B_n$, but $x_n \notin B_{n+1}$. Since $X\backslash U$ is countably compact, there is a cluster point, say y, of $\{X_n\}_{n=1}^{\infty}$ in $X\backslash U$. Now since all but finitely many points of the sequence are in $B_n, y \in B_n$ for any n. Let $B \in \mathcal{B}$ such that $y \in B$ and $x \notin B$. This means that $B_n \notin B$ For each n. Hence $B \subset B_n$ for all n , and this contradicts the assumption that y is a cluster point of $\{x_n\}_{n=1}^{\infty}$. Thus only a finite number of members of $B(x)$ intersect $X\backslash U$.

Corollary 4.5 *Let X be a compact space. Then X is a n. - a. space if and only if X has a countable base of clopen sets.*

Corollary 4.6 *Let X be a locally compact space. Then X is a n. - a.space if and only if X is n. - a. metrizable.*

The following example illustrates a non-Archimedean space that is not metrizable, and therefore not non-Archimedean metrizable.

Let X be the set of real numbers $\mathbb{R}$. Define a topology on $\mathbb{R}$ that has the following base. Let $B_{k,n} = \left(\alpha + \frac{k}{2^n}, \alpha + \frac{k+1}{2^n}\right)$ where $k \in \mathbb{Z}, n \in \mathbb{N}$ and α be a fixed irrational number. Let $\mathcal{B} = \{\{x\}: x \text{ irrational}\} \cup \{B_{k,n}\}_{\substack{k\in\mathbb{N} \\ n\in\mathbb{N}}}$. To show that $\mathcal{B}$ is a $n.-a.$base of X. Let $x \in X$, if x is an irrational number, then $\{x\} \in \mathcal{B}$, if x is a rational number then $x \neq \alpha + \frac{k}{2^n}$ for each $k \in Z$, $n \in \mathbb{N}$. So there exist $l \in \mathbb{Z}, m \in \mathbb{N}$ such that $x \in \left(\alpha + \frac{l}{2^m}, \alpha + \frac{l+1}{2^m}\right) \in \mathcal{B}$. Now, if $B_{k,n}, B_{l,m} \in \mathcal{B}$ then:

Case 1) if $n = m$ and k, l are any two integers such that $k < l$ then the intervals $B_{k,n} = \left(\alpha + \frac{k}{2^n}, \alpha + \frac{k+1}{2^n}\right)$ and $B_{l,n} = \left(\alpha + \frac{l}{2^n}, \alpha + \frac{l+1}{2^n}\right)$ are disjoint, since $\alpha + \frac{k+1}{2^n} \leq \alpha + \frac{l}{2^n}$. To show this, suppose not, so $\frac{l}{2^n} < \frac{k+1}{2^n}$ then $l < k + 1$ implies that $l - k < 1$, contradiction.

Case 2) If $k = 0$ and n, m are any natural numbers, where $n < m$ then $B_{k,m} \subset B_{k,n}$.

Case 3) If $k=-1$ and $n,m\in\mathbb{N}$ where $n<m$ then $B_{k,m}\subset B_{k,n}$.
Case 4) If k is a fixed negative number less than -1 and $n,m\in\mathbb{N}$, where $m=n+1$ then: $B_{k,n}, B_{k,m}$ are disjoint intervals. Since, $\alpha+\frac{k+1}{2^n}=\alpha+\frac{k+1}{2^{m-1}}=\alpha+\frac{(k+1)2}{2^m}\leq\alpha+\frac{k}{2^m}$.
Case 5) If k is a fixed positive integer number and $n,m\in\mathbb{N}$ where $m=n+1$ then $B_{k,n}, B_{k,m}$ are disjoint intervals and $\alpha+\frac{k+1}{2^m}\leq\alpha+\frac{k}{2^n}$. To show this, suppose not, so $\alpha+\frac{k}{2^n}<\alpha+\frac{k+1}{2^m}$ this implies $\frac{2k}{2^n}<\frac{k+1}{2^n}$. Hence $2k<k+1$, so $k<1$, a contradiction. In general, for any natural numbers n,m and any integers k, l, we have, if

$$\frac{k}{2^n}<\frac{l}{2^m}<\frac{k+1}{2^n}\ldots\ldots\ldots\ldots.(1)$$

then, $\frac{l+1}{2^m}\leq\frac{k+1}{2^n}$. To show this, suppose not, this is mean

$$\frac{k+1}{2^n}<\frac{l+1}{2^m}\ldots\ldots\ldots\ldots\ldots(2)$$

If $n>m$, then we have a contradiction with (1).
If $m>n$, from (1) we have $2^{m-n}k<l<2^{m-n}(k+1)$, and from (2) we have $2^{m-n}(k+1)<l+1$, hence $l<2^{m-n}(k+1)<l+1$. Since $2^{m-n}(k+1)$ is an integer number, this gives a contradiction. Now the space X is $\mathrm{n.-a.}$ and the set $\mathbb{Q}$ of all rational numbers is closed in X , To show $\mathbb{Q}$ is not a $G_\delta-$ set in X . Suppose $\mathbb{Q}$ is G_δ - set in X, hence $\mathbb{Q}=\bigcap_{n\in IN}U_n$, U_n is open for each $n\in\mathrm{IN}$. Let $A_n=\{x\in\mathrm{IR}:x\notin U_n\}, n\in\mathrm{IN}$, so $\mathrm{IR}=(\bigcup_{n\in IN}A_n)\cup\left(\bigcup_{q\in Q}\{q\}\right)$. By the Baire category theorem see, IR cannot be a union of closed nowhere dense sets, so there exists $m\in\mathbb{N}$, such that A_m contains an interval, this means, there exist $a,b\in\mathbb{R}, a<b$ and $(a,b)\subset A_m$, a contradiction. Hence X is not perfectly normal, therefore, by Proposition 2.6, X is not metrizable, hence not non-archimedean metrizable.

4.2 Zero-dimensionality property

Zero-dimensionality property in topology characterizes spaces with bases consisting entirely of clopen sets. This property implies a high degree of separability, as clopen sets provide precise "building blocks" for the topology. Zero-dimensional spaces are always totally disconnected, meaning that their only connected subsets are singletons. Zero-dimensional spaces often arise in analysis and topology due to their structural simplicity and are closed under subspaces and finite or infinite product topologies. These spaces are crucial in areas like Stone duality, where they correspond to Boolean algebras, and in constructing counterexamples or specialized models within general topology.

Definition 4.7 *A space X is called zero-dimensional if X is a non-empty T_1 Space with a base consisting of clopen (open and closed) sets.*

The non-archimedean spaces are subclasses of zero-dimensional spaces. This follows from the definition of zero-dimensional space and Lemma 3.2.

The Sorgenfrey line $\mathbb{R}_S$, as mentioned above, is an example of zero-dimensional space that is not non-archimedean.

A non-Archimedean space has a basis made up entirely of clopen sets. This clopen structure has important implications for the large inductive dimension function, often denoted by Ind, due to the distinctive properties of these spaces.

The large inductive dimension function is defined in [7] as follows:

Definition 4.8 *Let X be a topological space. Then we assign the large inductive dimension of X, denoted by $\mathrm{Ind}(X)$, which is an integer larger than or equal to -1 or the "infinite number" ∞; the definition of the dimension function Ind consists of the following conditions:*

(BC1) $Ind(X) = -1$ if and only if $X = \phi$;

(BC2) $Ind(X) \leq n$, where $n = 0,1,2,\ldots$, if for every closed set $A \subseteq X$ and each open set $U \subseteq X$ which contains A , there exists an open set $V \subseteq X$ such that $A \subseteq V \subseteq U$ and $Ind(bd(V)) \leq n - 1$.

(BC3) $Ind(X) = n$ if and only if $Ind(X) \leq n$ and $IndX > n - 1$;

(BC4) $Ind(X) = \infty$ if $Ind(X) > n$ for $n = -1,0,1,\ldots$

The large inductive dimension Ind is also called the Brouwer-Cech dimension. If $Ind(X)$ is defined then $Ind(F)$ is defined for every closed subspace F of X.

Proposition 4.9 *For any n. - a. space X, we have $Ind(X) = 0$.*

Proof. Let X be a $n.-a.$ space and $\mathcal{B}$ be a $n.-a.$ base of X . Let A, F be disjoint closed sets in X.

Then we have $U = \cup\{B_\alpha \in \mathcal{B}: B_\alpha \cap F = \emptyset, B_\alpha \cap A \neq \emptyset\}$ and $V = \cup\{B_\alpha \in \mathcal{B}: B_\alpha \cap A = \emptyset, B_\alpha U\}$ are two clopen sets of X separating A and F. Since, if $U \cap V \neq \emptyset$, then there exist $z \in U \cap V$ and $B_{\alpha_1}, B_{\alpha_n} \in B$ such that $z \in B_{\alpha_1} \cap B_{\alpha_n}$ where $B_{\alpha_1} \subset U, B_{\alpha_n} \subset V$. Since $B_{\alpha_1} \subset U$, so $B_{\alpha_1} \cap F = \emptyset$ and $B_{\alpha_1} \cap A \neq \emptyset$. Since $B_{\alpha_n} \subset V$, implies that $B_{\alpha_n} \cap A = \emptyset$, $B_{\alpha_n} U$. Since B is a n. - a. base of X, so either $B_{\alpha_1} \subset B_{\alpha_n}$, in this case, $B_{\alpha_1} \subset V$ and then $B_{\alpha_1} \cap A = \emptyset$. Which gives a contradiction. Or $B_{\alpha_n} \subset B_{\alpha_1}$, so $B_{\alpha_n} \subset U$. Also gives a contradiction. Hence $U \cap V = \emptyset$.

Now, if $a \in A$ then there exists $B_\beta \in \mathcal{B}$ such that $a \in B_\beta, B_\beta \cap F = \emptyset$. So we have $B_\beta \subset U$, hence $a \in U$, that is mean $A \subset U$. Similarly, if $y \in F$, there exists $B_{\alpha_n} \in \mathcal{B}$ such that $y \in B_{\alpha_n}, B_{\alpha_n} \cap A = \emptyset$, therefore, $B_{\alpha_n} \subset V$. So $y \in V$, that is, $F \subset V$.

The following theorem plays a significant role in dimension theory, serving as a tool for classifying and analyzing zero-dimensional spaces. It is closely associated with a unique property of non-Archimedean metrizable spaces and the characterization of spaces with large inductive dimension zero *Ind(X)=0*.

Theorem 4.10 *A metric space X is $n.-a.$ metrizable if and only if $Ind(X) = 0$.*

Proof. ($\Rightarrow$) Let X be a metric space and n. - a. metrizable, so X is $n.-a.$ space, and by Proposition 4.9, we have $Ind(X) = 0$.

($\Leftarrow$) Let X be a metric space with $Ind(X) = 0$, let H be an open set in X, so H may be considered as a union of a countable number of mutually disjoint clopen sets say $H = \bigcup_{n=1}^{\infty} U_n$. To prove this, let d be a metric on X and let $V_{\frac{1}{n}}(n \in IN)$ be a neighbourhood of H^c. Let $U_1, U_2, \dots, U_{n-1}$ be disjoint clopen subsets of H are already defned. We now, proceed to define $U_n \subset H$ in the following way:

Let $A = X \backslash H$, $B = \left(X \backslash V_{\frac{1}{n}}\right) \cup (\bigcup_{i=1}^{n-1} U_i)$ be disjoint closed sets. Since Ind $X = 0$, so by theorem (3.1.9) A and B can be separated by clopen sets. Let T be clopen set containing B and contained in H , then let $U_n = T \setminus \bigcup_{i=1}^{n-1} U_i$. In this way, any $H_{n\alpha}$ is decomposed into a countable number of disjoint sets $U_{in\alpha} (i = 1,2, \dots)$. The countable number of families $\{U_{in\alpha}\} = \{U_{k\alpha}\}, (k = 1,2, \dots)$ originated by enumerating the pairs $k = (i, n)$ is locally finite clopen base of X, so by theorem (4.2.7) X is n. - a. metrizable.

Conclusion. This paper has studied the connection between non-Archimedean bases, non-Archimedean metrizability, and zero-dimensional spaces. It has shown that the presence of a non-Archimedean base serves as both a necessary and sufficient condition for a space to be non-Archimedean metrizable. Moreover, the study demonstrated that the structure of a non-Archimedean base, defined by its clopen (simultaneously open and closed) sets, offers a natural framework for understanding the topological and metric properties of zero-dimensional spaces. Future research into these properties will further deepen our understanding of their interconnections.

References:

[1] Alexandroff, P., (1960), on the metrization of topological spaces. Bull. Acad. Sci. 8, $135 - 140$.

[2] Artico G., Marconi U., and Moresco R., (2007), Selectors in non-archimedean spaces, Topology and its Applications, vol. 154, 540-551.

[3] Charalambous, Michael G., (2019), Dimension Theory: A selection of theorems and counterexamples, Atlantis studies in Mathematics.

[4] Conover, R. A., (2014), A First course in topology an introduction to mathematical thinking, Manufactured in the US by Coutier corporation 78001501

[5] Ellis R., (1970), Extending continuous functions on zero-dimensional spaces. Math. Ann. 186, 114 - 122.

[6] Engelking, R., (1989). “General topology”, Second ed. Sigma Series in pure mathematics, vol. 6, Heldermann Verlag, Berlin, Translated from Polish by the author. MR 1039321 (91c:34001).

[7] Engilking R., (1978), Dimension Theory, Warszawa.

[8] Nagata J., M., (1984), Dimension Theory, Amsterdam.
[9] Nyikos P. and Reichel H. C., (1975), on the structure of zero-dimensional spaces. Indag. Math. 37, 120 - 136.
[10] Pinter, C. C., (2014), A book of set theory, Dover publications, inc. Mineola, New York
[11] Hodel, R. E., (2012), Nagata's research in dimension theory, Topology and its Applications, 159, 1545-1558.
[12] Viro, O. Y., Ivanov, O. A., Netsvetaev, N. Y., Kharlamov, V. M. (2008), Elementary topology problem textbook, AMS Providence Rhode Island.